Faculty of Science, Technology, Engineering and Mathematics
M303 Further pure mathematics

M303

TMA 06 2020J

Mainly covers Book F *Metric spaces 2*
Cut-off date 4 May 2021

You will find instructions for completing TMAs in the 'Assessment' area of the M303 website. Please remind yourself of these instructions before beginning work on this TMA. You can submit this TMA either by post, together with a completed TMA form (PT3), to arrive by the cut-off date given above, or electronically, as a single PDF file, using the University's online TMA/EMA service.

Half of the TMA is formative, and the other half is summative. The formative questions are designed to either *extend* your understanding of the topics or *revise* topics from earlier books, while the summative questions place more emphasis on *assessing* your understanding of the topics. Both types of question will be marked by your tutor, but (unlike the summative questions) marks for the formative questions do not count towards your final grade. You should submit your solutions to both sets of questions to your tutor by the cut-off date.

The marks allocated to each part of a question are indicated in the margin.

Although many of the questions on this assignment require numerical answers, the questions invariably carry *method marks*.

In phrasing the questions, we have used the words listed below, each of which should be interpreted as indicated.

prove, show, explain, justify	Clear reasoning and explanation for all steps are called for.
determine, find, devise, calculate, compute	An indication of the method used and all working in arriving at an answer should be given.
deduce	Clear explanation of how one result follows from another is required.
solve	Working must be shown. A numerical answer alone is not sufficient.
evaluate, give, write down, list	Answer suffices. No explanation need be given.
hence	No marks will be awarded for any alternative method.

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This TMA is based on Book F (Metric spaces 2).

Part A is **summative**. These questions assess your knowledge of the module. You must not discuss these questions in the forums, and your tutor is not allowed to help you directly.

Part B is **formative**. We recommend that you submit two of these questions to your tutor for feedback.

Doing the formative questions will help you to understand key concepts covered in the text. You are encouraged to discuss these questions in the module forums and with your tutor (who is allowed to help you).

Questions 6, 7 and 8 cover material that is more central to the module, while Questions 9 and 10 are designed to expand your understanding beyond the core material.

Your tutor will mark all of the answers that you submit. But note that only your marks for Part A count towards your overall assessment score.

Some of these questions ask you for a sketch or a description of a diagram. When sketching or describing, you should follow the conventions used in the module text. Take care to indicate which points are included: use or describe dashed lines to indicate boundaries that are not included, and solid lines to indicate boundaries that are. Use or describe small open circles (○) to indicate any 'corner' points on a boundary that are not included, and small closed circles (●) to indicate any 'corner' points that are included.

Part A Summative

Question 1 – 10 marks

This question tests your understanding of material covered in Chapter 21.

Let d be the Euclidean metric on $\mathbb{R}^2$ given by

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

Consider the following four sets in $\mathbb{R}^2$:

$$W = \{(x, y) \in \mathbb{R}^2 : x < 3\} \cup \{(x, y) \in \mathbb{R}^2 : x > 3 \text{ and } x < 5\},$$
$$X = \{(2, y) \in \mathbb{R}^2 : y \in \mathbb{R}\} \cup \{(x, 1) \in \mathbb{R}^2 : x \in \mathbb{R}\},$$
$$Y = \left\{ \left(\frac{1}{n}, \frac{1}{n}\right) \in \mathbb{R}^2 : n \in \mathbb{Z} - \{0\} \right\},$$
$$Z = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 2)^2 = 4\} \cup \{(x, y) : x^2 + (y + 2)^2 < 4\}.$$

For each set, do the following.

(a) On a separate diagram, sketch the set, by hand or otherwise. (If you are unsure how to represent something on a diagram, or are unable to sketch it, then describe what you mean in words. You are also free to use a combination of sketching and writing.) [*4*]

(b) Write down whether the set is d-connected or d-disconnected. If the set is d-disconnected, then write down a d-disconnection. [*6*]

Question 2 – 8 marks

This question tests your understanding of material covered in Chapter 22.

Let

$$P = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 4\},$$

and let d be the Euclidean metric on $\mathbb{R}^2$ given by

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

(a) Sketch P, by hand or otherwise. (If you are unable to sketch it for any reason, then describe what you mean in words. You are also free to use a combination of sketching and writing.) [*1*]

(b) Write down a finite d-open cover of P. [*1*]

(c) Write down an infinite d-open cover of P that has a finite subcover. [*1*]

(d) Write down a finite subcover for your answer to part (c). [*1*]

(e) Write down an infinite d-open cover of P that does not have a finite subcover. [*2*]

(f) Part (e) demonstrates that P is not d-compact. Explain why this does not contradict Theorem 5.14 of Chapter 22. [*2*]

Question 3 – 3 marks

This question tests your understanding of material covered in Chapter 23.

Let (a_n) be the sequence given by

$$a_n = 1 - \frac{1}{n}, \quad \text{where } n \in \mathbb{N}.$$

(a) Write down an example of a set $X_1 \subseteq \mathbb{R}$ that contains all points of the sequence (a_n), and a metric $d_1 \colon X_1 \times X_1 \to \mathbb{R}$, such that (a_n) is a d_1-convergent d_1-Cauchy sequence. [*1*]

(b) Write down an example of a set $X_2 \subseteq \mathbb{R}$ that contains all points of the sequence (a_n), and a metric $d_2 \colon X_2 \times X_2 \to \mathbb{R}$, such that (a_n) is not a d_2-Cauchy sequence. [*1*]

(c) Write down an example of a set $X_3 \subseteq \mathbb{R}$ that contains all points of the sequence (a_n), and a metric $d_3 \colon X_3 \times X_3 \to \mathbb{R}$, such that (a_n) is a d_3-Cauchy sequence that is not d_3-convergent. [*1*]

Question 4 – 13 marks

This question tests your understanding of material covered in Chapters 21–23.

(a) Let d' be the metric on $\mathbb{R}$ given by

$$d'(x, y) = 2|x - y|, \quad \text{for } x, y \in \mathbb{R}.$$

Let $X \subseteq \mathbb{R}$ be given by

$$X = \{0\} \cup \left\{ \frac{1}{5^n} : n \in \mathbb{N} \right\},$$

and let $d = d'_X$.

(i) Prove that X is totally d-disconnected. [5]

(ii) (1) Prove that $X - B_d(0, r)$ is a finite set for each $r > 0$. [2]

(2) Hence, or otherwise, show that X is d-compact. [2]

(iii) Is X d-complete? Justify your answer. [1]

(b) Let e' be the metric on $\mathbb{R}^2$ given by

$$e'\big((x_1, y_1), (x_2, y_2)\big) = 2|x_1 - x_2| + d_0(y_1, y_2),$$

where d_0 is the discrete metric, and let $Y \subseteq \mathbb{R}^2$ be given by

$$Y = \{(0, 0)\} \cup \left\{ \left(0, \frac{1}{5^n}\right) : n \in \mathbb{N} \right\}.$$

Let $e = e'_Y$. Is Y e-compact? Justify your answer. [3]

Question 5 – 16 marks

This question tests your understanding of material covered in Chapter 24.

(a) Let

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\},$$
$$B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 9\} \cup \{(x, 0) \in \mathbb{R}^2 : 3 \leq x \leq 6\}.$$

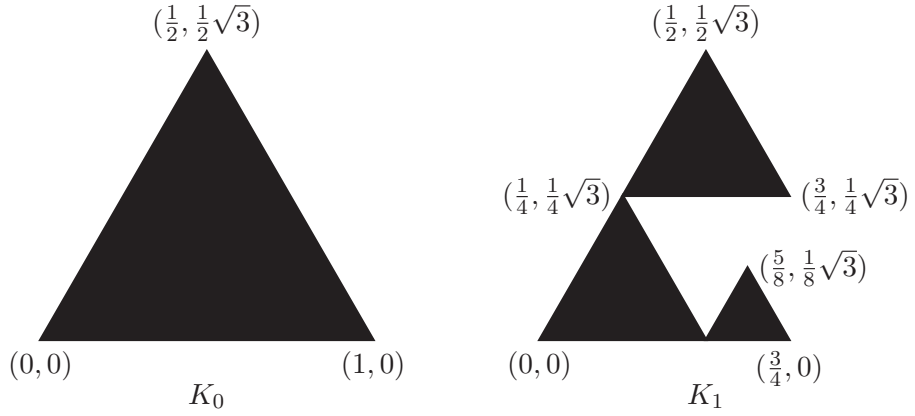
Let $d^{(2)}$ be the Euclidean distance between compact sets (see Definition 2.7 of Chapter 24, page 210 of Book F).

(i) Sketch A and B. (If you are unable to do a sketch for any reason, then describe what you mean in words. You are also free to use a combination of sketching and writing.) [1]

(ii) (1) Find $d^{(2)}(A, B)$ and $d^{(2)}(B, A)$. [2]

(2) Hence, or otherwise, determine $d_H(A, B)$, the Hausdorff distance between the sets A and B. [1]

(b) Let K_0 and K_1 be as shown below.



The figure shows two diagrams next to one another. On the left, the diagram consists of one filled in equilateral triangle with sides of length 1. A horizontal base goes from $(0,0)$ to $(0,1)$, and the apex of this triangle is labelled as $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$. This diagram is labelled K_0.

On the right, the diagram consists of three filled in equilateral triangles. The first has side length a half, and a horizontal base going from the point $(0,0)$ to the point $(\frac{1}{2}, 0)$. The second triangle has side length a half, and a horizontal base going from the point $(\frac{1}{4}, \frac{1}{4}\sqrt{3})$ to the point $(\frac{3}{4}, \frac{1}{4}\sqrt{3})$. The apex of this uppermost triangle is labelled $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$. The third triangle has side length a quarter, and a horizontal base going from the point $(\frac{1}{2}, 0)$ to the point $(\frac{3}{4}, 0)$. The apex of the third triangle is at $(\frac{5}{8}, \frac{3}{8}\sqrt{3})$. This diagram is labelled K_1.

(i) Write down three similarities $S_1, S_2, S_3 \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$K_1 = \bigcup_{i=1}^{3} S_i(K_0),$$

and state their similarity ratios. *[3]*

(ii) Let

$$K_{n+1} = \bigcup_{i=1}^{3} S_i(K_n), \quad \text{for } n \in \mathbb{N}.$$

There are triangles of three different widths in K_2. How many triangles are there in K_2 of the following widths?

(1) $\frac{1}{4}$ (2) $\frac{1}{8}$ (3) $\frac{1}{16}$ *[3]*

(iii) Let

$$K = \bigcap_{n=1}^{\infty} S^n(K_0),$$

where $S \colon \mathcal{K}(\mathbb{R}^2) \to \mathcal{K}(\mathbb{R}^2)$ is given by

$$S(F) = \bigcup_{i=1}^{3} S_i(F).$$

Prove that K is the non-empty compact invariant set of the iterated function scheme defined by S_1, S_2 and S_3. *[2]*

(iv) Hence determine $\dim(K)$. *[4]*

Part B Formative

Question 6 – 7 marks

This question revises material from Book D.

Let $d\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be the distance function on $\mathbb{N}$ given by

$$d(i,j) = \left| \frac{i}{i+1} - \frac{j}{j+1} \right|.$$

Prove that d is a metric. [7]

Question 7 – 13 marks

This question revises material from Book D.

(a) Let

$$A = \left\{ \frac{1}{p} \in \mathbb{R} : p \in \mathbb{N} \text{ is prime} \right\},$$

and let d be the Euclidean metric on $\mathbb{R}$.

(i) Determine whether A is d_A-open, d_A-closed, both d_A-open and d_A-closed, or neither d_A-open nor d_A-closed. [2]

(ii) Determine whether A is d-open, d-closed, both d-open and d-closed, or neither d-open nor d-closed. [2]

(iii) Write down the d-interior and the d-closure of A. [2]

(b) For each of the following, determine whether the set is d-open, d-closed, both d-open and d-closed, or neither d-open nor d-closed, as a subset of $\mathbb{R}^2$, where d is the usual Euclidean metric on $\mathbb{R}^2$.

(i) $X = \{(x,y) \in \mathbb{R}^2 : |x| \leq 1,\ |y| \leq 100 \text{ and } y \in \mathbb{Z}\}$ [2]

(ii) $X = \{(x,y) \in \mathbb{R}^2 : |x| \leq 1 \text{ and } y \in \mathbb{Z}\}$ [2]

(iii) $X = \{(x,y) \in \mathbb{R}^2 : |x| \leq 1 \text{ and } y \in \mathbb{Q}\}$ [3]

Question 8 – 8 marks

This question allows you to practise techniques covered in Chapter 22.

Let

$$A = \{f \in C[0,1] : |f(x)| \leq \exp(x) \text{ for each } x \in [0,1] \\ \text{and } |f(x) - f(y)| \leq \sqrt{|x-y|} \text{ for } x, y \in [0,1]\}.$$

Show that A is $d_{\max}$-sequentially compact. [8]

Question 9 – 10 marks

This question develops your understanding of material covered in Chapter 23.

In Exercise 2.2 of Chapter 15, we introduced the metric space of eventually zero sequences (ℓ_0, d), where

$$\ell_0 = \{(a_n) : a_n \in \mathbb{R} \text{ for each } n, \text{ and } (a_n) \text{ is eventually zero}\}$$

and $d\colon \ell_0 \times \ell_0 \to \mathbb{R}$ is given by

$$d(\mathbf{a}, \mathbf{b}) = \sum_{n=1}^{\infty} |a_n - b_n|, \quad \text{where } \mathbf{a} = (a_n) \text{ and } \mathbf{b} = (b_n).$$

In this question, we investigate (ℓ_0, d) further.

(a) Let p be a prime and let $(\mathbf{a}_k)$ be the sequence of points in ℓ_0 given by

$$\mathbf{a}_k = \left(\frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}, \ldots, \frac{1}{p^k}, 0, 0, 0, \ldots\right), \quad \text{for } k \in \mathbb{N}.$$

(i) Calculate $d(\mathbf{a}_{10}, \mathbf{a}_9)$. [*1*]

(ii) Find $d(\mathbf{a}_{k+1}, \mathbf{a}_k)$ for each $k \in \mathbb{N}$. [*2*]

(iii) Hence show that $(\mathbf{a}_k)$ is a Cauchy sequence. [*4*]

(b) Use your results from part (a) to prove that (ℓ_0, d) is not a complete metric space. [*3*]

Question 10 – 12 marks

This question is designed to help you to revise some topics from earlier books, and takes a second look at the metric on a group introduced in Question 11 of TMA 04. You do not need to have attempted Question 11 of TMA 04 to be able to do this question.

Let G be a group, let $n \in \mathbb{N}$, and suppose that $H_0, H_1, H_2, \ldots, H_n$ are subgroups of G such that

$$\{e\} = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = G,$$
with $H_i \neq H_{i-1}$ for $i = 1, 2, \ldots, n$.

We define a metric $d_G\colon G \times G \to \mathbb{R}$ by

$$d_G(g, h) = k, \quad \text{where } gh^{-1} \in H_k \text{ and } gh^{-1} \notin H_i \text{ for all } i < k.$$

(You may assume that d_G is a metric for G.)

(a) (i) For $g \in G$, describe $B_{d_G}(g, 1)$, $B_{d_G}[g, 1]$ and $S_{d_G}(g, 1)$.

(ii) For $g \in G$, describe $B_{d_G}(g, 2)$, $B_{d_G}[g, 2]$ and $S_{d_G}(g, 2)$. [*7*]

(b) Describe the d_G-closed and d_G-open sets of G. [*2*]

(c) What conditions, if any, are necessary and sufficient for a group G to be each of the following?

(i) d_G-connected

(ii) d_G-compact

(iii) d_G-complete [*3*]